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# Integer-ambiguity resolution in astronomy and geodesy

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Recent theoretical developments in astronomical aperture synthesis have revealed the existence of integer-ambiguity problems. Those problems, which appear in the self-calibration procedures of radio imaging, have been shown to be similar to the nearest-lattice point (NLP) problems encountered in high-precision geodetic positioning, and in global navigation satellite systems. In this paper, we analyse the theoretical aspects of the matter and propose new methods for solving those NLP problems. The related optimization aspects concern both the preconditioning stage, and the discrete-search stage in which the integer ambiguities are finally fixed. Our algorithms, which are described in an explicit manner, can easily be implemented. They lead to substantial gains in the processing time of both stages. Their efficiency was shown via intensive numerical tests.

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## 1 Introduction

Astronomical images obtained from ground-based observatories are degraded by atmospheric turbulence. In particular, the phase of the Fourier transform of the object-source distribution is severely perturbed which leads to a significant loss of angular resolution in the resulting images. Thanks to the theoretical and technical developments of the last half century, large interferometric arrays circumvent this difficulty in radio-astronomy, and now routinely provide sharp-edged images with a very high angular resolution.

One of the methods used for obtaining those nice results is ‘self-calibration.’ In the most general case, the vectorial nature of the electromagnetic field must be taken into account in the very formulation of the problem; see Hamaker (2000) and references therein. In this paper, we however restrict ourselves to ‘scalar self-calibration.’ We thus refer to the same framework as that defined in Lannes & Prieur (2011). In particular, we show that in the scalar case, the phase-calibration problem has a close similarity with the calibration problems encountered in high-precision geodetic positioning and in global navigation satellite systems (GNSS).

In fact, the approach we propose for solving the phase-calibration problem in the scalar case is a good starting point for tackling the more complex problem of full polarimetric phase calibration. This possible extension however deserves a particular analysis which goes beyond the scope of the present paper. Some guiding ideas for the corresponding ‘matrix self-calibration’ approach are to be found

in Hamaker (2000) and Yatawatta (2012). The scalar case presented in Sect. 2.1 has already its own complexity. Any vectorial analysis should therefore start from a good understanding of that analysis. We intend to address the global problem in a forthcoming paper.

In a previous paper (Lannes & Prieur, 2011), we have analysed the self-calibration procedure in the scalar case. In that special case, we have proposed a new approach to the problem: the ‘arc-approach.’ The final step of that approach consists in solving a nearest-lattice-point (NLP) problem; for a precise definition of this problem, see Sect. 3.1.

In fact, NLP problems appear in many fields of applied mathematics. In particular, as already mentioned, they play a central role in high-precision geodetic positioning and in GNSS; see, e.g., Lannes & Prieur (2013). In this paper, we present new methods for solving those NLP problems. These methods can therefore be applied both in astronomy and geodesy.

The standard way of solving an NLP problem includes two stages: a preconditioning stage, and a discrete-search stage in which the integer ambiguities are finally fixed. The problem is usually preconditioned by implementing the algorithm introduced by Lenstra, Lenstra & Lovász (1982): the LLL algorithm. The LAMBDA decorrelation method of Teunissen (1995) can also be used for this purpose; for the theoretical link between LLL-reduction and LAMBDA-decorrelation, see Lannes (2013). The NLP problem is then solved in the reduced basis thus obtained. This is done via appropriate discrete-search techniques. In this general context, we present our implementation of the LLL algorithm, as well as our discrete-search techniques. This pa-

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per thus revisits and completes the appendix 3 of Lannes & Prieur (2011). With regard to the current state of the art (see, e.g., Agrell et al. (2002), Jazaeri et al. (2012)), the methods described in this paper lead to a speed-up of the order of two.

In Sect. 2, we show how those problems appear in astronomy and geodesy. The main guidelines of our study are presented in Sect. 3. Some basic notions are then defined among which that of LLL-reduced basis. In Sect. 4, we then describe an LLL-type algorithm allowing an LLL-reduced basis to be built. Section 5 is devoted to the main contribution of our paper: the discrete-search techniques to be implemented for finding the nearest lattice point in the selected reduced basis. We also describe the techniques to be used for identifying the points lying in some neighbourhood of the nearest lattice point. Indeed, these points are also useful for the analysis of the related problems. The computational issues of our contribution and its main results are summarized in Sects. 6 and 7.

## 2 NLP problems in astronomy and geodesy

We here present some NLP problems encountered in astronomy (Sect. 2.1) and geodesy (Sect. 2.2). The similarities between the scalar case presented in Sect. 2.1, and the global positioning problems of Sect. 2.2 are thus explicitly exhibited.

### 2.1 Self-calibration procedures in phase-closure imaging

When mapping incoherent sources with aperture-synthesis devices, the pupil-phase perturbations (hereafter pupil-phase biases) caused by the atmospheric turbulence degrade the angular resolution of the restored image. A standard way for obtaining high angular-resolution images is to estimate those pupil-phase biases from observations of a calibrator (usually a reference star). However when the turbulence is strong and quickly varies with time, this procedure is not possible. A way out is to use ‘self-calibration’ which corresponds to the situation where the object source to be imaged plays the role of the calibrator. Following the pioneering work of Cornwell & Wilkinson (1981) in the scalar case, this problem can then be solved by alternate phase-calibration operations and Fourier-synthesis processes. However, this procedure is generally rather unstable. To ensure the reliability and the robustness of those techniques, the phase-calibration operations must then be conducted with much care.

The model of the object source is refined throughout the iterative self-calibration procedure. At each iteration, the phase-calibration operation consists in estimating virtual pupil-phase biases  $\alpha_d(i)$  so that the following equation is satisfied in a least-squares sense to be defined:

$$\exp i\varphi_d(i, j) \exp -i[\alpha_d(i) - \alpha_d(j)] = \exp i\varphi_m(i, j) \quad (1)$$

Here,  $\exp i\varphi_d$  and  $\exp i\varphi_m$  are the ‘phasors’ of the (complex) ‘visibility functions’ of the data and the model, respectively. The pairs  $(i, j)$ , which define the edges of the ‘phase-calibration graph’  $\mathcal{G}$ , correspond to the baselines of the interferometric device; for further details, see Sect. 2 in Lannes & Prieur 2011. The self-calibration procedure aims at reducing the phase discrepancy

$$\varphi \stackrel{\text{def}}{=} \varphi_d - \varphi_m \quad (2)$$

From Eq. (1), we infer that the phase-calibration operation consists in finding a function  $\alpha_d$  such that the following relationship be valid up to error terms:

$$\varphi(i, j) - [\alpha_d(i) - \alpha_d(j)] = 2\pi N(i, j) \quad (3)$$

with  $N(i, j)$  in  $\mathbb{Z}$ . In radio-astronomy, the related optimization problems are generally solved at the phasor level: one minimizes the size of the chords associated with the phasors  $\exp i\{\varphi(i, j) - [\alpha_d(i) - \alpha_d(j)]\}$

In some critical situations, the ‘chord functional’ may have several minima. As shown in Lannes (2005), and Lannes & Prieur (2011), the analysis of the problem must then be conducted at the phase level. We then consider the size of the quantities

$$\arccos\{\varphi(i, j) - [\alpha_d(i) - \alpha_d(j)]\}$$

where function  $\arccos$  is defined as follows:

$$\arccos(\theta) \stackrel{\text{def}}{=} \theta - 2\pi \left\lfloor \frac{\theta}{2\pi} \right\rfloor$$

Here,  $\lfloor x \rfloor$  denotes the integer of  $\mathbb{Z}$  closest to  $x$ ; when  $x = k + 1/2$  for some  $k$  in  $\mathbb{Z}$ ,  $\lfloor x \rfloor$  is set equal to  $k$ . The functional to be minimized is then of the form

$$g_d(\alpha_d) \stackrel{\text{def}}{=} \|\arccos(\varphi - B\alpha_d)\|_w \quad (4)$$

where

$$(B\alpha_d)(i, j) \stackrel{\text{def}}{=} \alpha_d(i) - \alpha_d(j) \quad (5)$$

with  $\alpha_d(1) = 0$  for instance; the norm  $\|\cdot\|_w$  is defined as specified in Sect. 2.2 of Lannes & Prieur 2011.

As explicitly shown in Sects. 4 to 7 of that paper, the arc approach gives a better insight into the problem. The corresponding theoretical framework appeals both to algebraic graph theory (Biggs 1996) and algebraic number theory (Cohen 1996). We now give a survey of the matter which shows how those two main features are tightly imbricated.

The notion of ‘phase closure,’ which underlies the concept of ‘phase-closure imaging’ (PCI), is introduced in a context more general than that usually defined in radio imaging and optical interferometry. In particular, closure phases of order larger than three may then be defined. According to our algebraic-graph analysis, the data-model discrepancy can be decomposed in the form (see Sect. 3.3 of Lannes & Prieur 2011)

$$\varphi = \varphi_b + \varphi_c$$

Here, the baseline-bias function  $\varphi_b$  is equal to  $B\alpha^{(\varphi)}$  for

some  $\alpha^{(\varphi)}$  (depending on  $\varphi$ ). The function  $\varphi_c$  is the ‘closure function’ of  $\varphi$ ; it takes its values on the  $n_c$  ‘closure edges’ of  $\mathcal{G}$ , the ‘loop-entry baselines’ of the problem; see for example Figs. 3 and 4 in Lannes & Prieur (2011).

Clearly,  $\varphi - B\alpha_d = \varphi_c - B(\alpha_d - \alpha^{(\varphi)})$ . It then follows from Eq. (4) that  $g_d(\alpha_d)$  is equal to  $g(\alpha)$  where

$$g(\alpha) \stackrel{\text{def}}{=} \|\text{arc}(\hat{\varphi}_c - B\alpha)\|_w \quad (6)$$

with

$$\alpha \stackrel{\text{def}}{=} \alpha_d - \alpha^{(\varphi)} \quad \text{and} \quad \hat{\varphi}_c \stackrel{\text{def}}{=} \text{arc}(\varphi_c) \quad (7)$$

The minimizers of  $g_d$  can therefore be easily deduced from those of  $g$ .

Divided by  $2\pi$ ,  $\hat{\varphi}_c$  defines some point  $\hat{\mathbf{v}}$  of  $\mathbb{R}^{n_c}$ . We have shown that the minima of the arc functional  $g$  are determined via particular ‘integer sets’ associated with  $\hat{\mathbf{v}}$ . Those integer sets correspond to some particular points  $\tilde{\mathbf{v}}$  of lattice  $\mathbb{Z}^{n_c}$ ; see Property 2 in Lannes & Prieur 2011. In that algebraic-number framework, finding the global minimizer of  $g$  (and thereby that of  $g_d$ ) amounts to finding the point  $\tilde{\mathbf{v}}$  of  $\mathbb{Z}^{n_c}$  closest to  $\hat{\mathbf{v}}$  with regard to some distance; that distance is defined via some quadratic form whose matrix  $\mathbf{Q}$  is the inverse of the variance-covariance matrix  $\mathbf{V}$  of  $\hat{\mathbf{v}}$ . As explicitly clarified in Sect. 3.1, finding the global minimum therefore amounts to solving a NLP problem in which  $\hat{\mathbf{v}}$  can be regarded as its ‘float solution.’ The main secondary minima of  $g$ , if any, correspond to  $\mathbb{Z}^{n_c}$ -lattice points in some neighbourhood of  $\tilde{\mathbf{v}}$ . Like for  $\hat{\mathbf{v}}$ , those points can be identified, in a systematic manner, via the integer-programming techniques presented in this paper.

## 2.2 High-precision geodetic positioning

The techniques involved in high-precision geodetic positioning and global navigation satellite systems (GNSS) are based on two types of data: the (carrier-)phase and code (or pseudo-range) observations; see, e.g., Lannes & Gratton (2009), Lannes & Teunissen (2011). The phase observational equations of GNSS networks are of the form

$$\begin{cases} b_\kappa(i, j) - [\beta_{r\kappa}(i) - \beta_{s\kappa}(j)] = N(i, j) \\ \text{for } \kappa = 1, \dots, k \end{cases} \quad (8)$$

In those problems,  $\kappa$  is the epoch index;  $k$  is the index of the current epoch;  $\beta_{r\kappa}(i)$  and  $\beta_{s\kappa}(j)$  are clock-phase biases. Those biases, which are expressed in cycles, depend on the frequency of the transmitted carrier wave; subscripts r and s stand for receiver and satellite,<sup>1</sup> respectively;  $i$  is the index of the receiver, and  $j$  that of the satellite;  $N(i, j)$  is the integer ambiguity of the corresponding carrier-phase measurement. The terms  $b_\kappa(i, j)$  include the corresponding phase data and the contributions associated with the real variables of the problem other than the clock-phase biases: position and atmospheric parameters, for instance; see, e.g., de Jonge

(1998) and Lannes & Teunissen 2011. The set of receiver-satellite pairs  $(i, j)$  involved in Eq. (8) forms the observational graph  $\mathcal{H}_\kappa$  of the GNSS scenario of epoch  $\kappa$ . Owing to the particular structure of the phase equations (8), the problem has a basic rank defect. As outlined below, the latter can be eliminated by an appropriate redefinition of its variables.

In the system of Eqs. (8), the GNSS functional  $N$  takes its values on  $\mathcal{G}$ , the union of the graphs  $\mathcal{H}_\kappa$  until the current epoch  $k$ . The similarity of Eqs. (3) and (8) was first pointed out by Lannes & Teunissen 2011. In Lannes & Prieur 2013, we were therefore led to propose for  $N$  a decomposition quite similar to that of  $\varphi$  in Sect. 2.1:  $N = N_b + N_c$  with  $N_b = B\mu^{(N)}$ . Here,  $\mu^{(N)}$  is an integer-valued function depending on  $N$ ;  $\mu^{(N)}$  takes its values on the vertices of  $\mathcal{G}$  other than the reference receiver  $r_1$  (for example). The corresponding ‘integer variable’  $\mathbf{v} := N_c$  lies in  $\mathbb{Z}^{n_c}$  where  $n_c$  is the number of closure edges of  $\mathcal{G}$ . The redefined clock-phase biases are then of the form  $\beta_{r\kappa}(i) + \mu_r^{(N)}(i)$  (for  $i \neq 1$ ) and  $\beta_{s\kappa}(j) + \mu_s^{(N)}(j)$ .

In a first stage, at each epoch  $k$ , the problem is solved in the LS sense by considering  $\mathbf{v}$  as a ‘float variable.’ A float solution  $\hat{\mathbf{v}}$  is thus obtained and updated progressively. In practice, this is done via recursive QR-factorization; see Appendix C in Lannes & Prieur 2013. The ambiguity solution  $\tilde{\mathbf{v}}$  is then the point of  $\mathbb{Z}^{n_c}$  closest to  $\hat{\mathbf{v}}$  with regard to some distance. Like in PCI, that distance is defined via some quadratic form whose matrix  $\mathbf{Q}$  is the inverse of the variance-covariance matrix  $\mathbf{V}$  of the float solution  $\hat{\mathbf{v}}$ . In that case, the points of  $\mathbb{Z}^{n_c}$  lying in some neighbourhood of  $\tilde{\mathbf{v}}$  are involved in the corresponding validation techniques; see Verhagen & Teunissen 2006. Again, like in PCI, those points can be identified, in a systematic manner, via the integer-programming techniques presented in this paper. Once  $\tilde{\mathbf{v}}$  has been fixed and validated, the real variables, among which the redefined clock-phase biases, are then estimated accordingly.

## 3 Guidelines

This paper is essentially devoted to the methods to be used for solving the NLP problems encountered in astronomy and geodesy. Setting  $n := n_c$ , we first define these problems as follows.

### 3.1 NLP problems

Given some vector  $\hat{\mathbf{v}}$  of  $\mathbb{R}^n$ , consider the (or a) vector  $\tilde{\mathbf{v}}$  of  $\mathbb{Z}^n$  such that

$$\tilde{\mathbf{v}} = \underset{\mathbf{v} \in \mathbb{Z}^n}{\text{argmin}} \|\mathbf{v} - \hat{\mathbf{v}}\|_{\mathbf{Q}}^2 \quad (9)$$

The norm introduced here is that of  $(\mathbb{R}^n, \mathbf{Q})$ : the space  $\mathbb{R}^n$  endowed with the inner product

$$(\mathbf{v} | \mathbf{v}')_{\mathbf{Q}} \stackrel{\text{def}}{=} (\mathbf{v} \cdot \mathbf{Q}\mathbf{v}') \quad (10)$$

where  $\mathbf{Q}$  is the inverse of the variance-covariance matrix of the ‘float solution’  $\hat{\mathbf{v}}$ :  $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{V}^{-1}$ . Clearly,  $(\cdot | \cdot)$  is the

<sup>1</sup> Here, satellite should be understood as satellite transmitter.

Euclidean inner product of  $\mathbb{R}^n$ . In matrix terms, we therefore have

$$(\mathbf{v} \mid \mathbf{v}')_{\mathbf{Q}} = \mathbf{v}^T \mathbf{Q} \mathbf{v}' \quad (11)$$

All the quantities appearing in these equations are expressed in the standard basis

$$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$

of  $\mathbb{R}^n$  and  $\mathbb{Z}^n$ . Note that this basis can be represented by the row matrix

$$\mathbf{B} \stackrel{\text{def}}{=} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_n] \quad (12)$$

whose entries are the vectors  $\mathbf{e}_j$  for  $j = 1, \dots, n$ .

The integer lattice  $\mathbb{Z}^n$  regarded as a subset of  $(\mathbb{R}^n, \mathbf{Q})$  is denoted by  $(\mathbb{Z}^n, \mathbf{Q})$ ;  $\hat{\mathbf{v}}$  is therefore a nearest lattice point to  $\mathbf{v}$  in  $(\mathbb{Z}^n, \mathbf{Q})$ . Equation (9) therefore defines an NLP problem.

### 3.2 Factorizations of $\mathbf{Q}$

In this paper, we write the Cholesky factorization of  $\mathbf{Q}$  in the form

$$\mathbf{Q} = \mathbf{R}^T \mathbf{R} \quad (13)$$

where  $\mathbf{R}$  is an upper-triangular matrix. Denoting by  $\|\cdot\|$  the Euclidean norm of  $\mathbb{R}^n$ , we therefore have, from Eqs. (9) and (11),

$$\hat{\mathbf{v}} = \underset{\mathbf{v} \in \mathbb{Z}^n}{\operatorname{argmin}} \|\mathbf{R}(\mathbf{v} - \hat{\mathbf{v}})\|^2 \quad (14)$$

Let  $\mathbf{D}$  be the diagonal matrix defined via the relation

$$\mathbf{R} = \mathbf{D}^{1/2} \mathbf{U} \quad (15)$$

where  $\mathbf{U}$  is an upper-triangular matrix whose diagonal elements  $u_{j,j}$  are equal to unity. For clarity, the diagonal entries of  $\mathbf{D}$  will be denoted by  $d_j$ . From Eq. (13), we have

$$\mathbf{Q} = \mathbf{U}^T \mathbf{D} \mathbf{U} \quad (16)$$

### 3.3 $\mathbf{Q}$ -Orthogonality defect

Any basis of  $\mathbb{Z}^n$  is characterized by a row matrix of the form

$$\mathbf{B} \stackrel{\text{def}}{=} [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_n] \quad (17)$$

In general, such a basis is far from being  $\mathbf{Q}$ -orthogonal; see Eq. (10). To provide a measure of this defect, we introduce the following notion.

**DEFINITION 3.1.** The parameter

$$\delta_{\mathbf{Q}}(\mathbf{B}) \stackrel{\text{def}}{=} \left( \frac{\prod_{j=1}^n \mathbf{e}_j^T \mathbf{Q} \mathbf{e}_j}{\det \mathbf{Q}} \right)^{1/(2n)} \quad (18)$$

is the ‘dilute  $\mathbf{Q}$ -orthogonality defect’ of  $\mathbf{B}$   $\square$

In the notation adopted in Eq. (18),  $\mathbf{e}_j$  denotes the column matrix whose entries are the components of the corresponding vector in the standard basis. Those entries therefore lie in  $\mathbb{Z}$ . Clearly,  $\det \mathbf{Q}$  is the determinant of  $\mathbf{Q}$ .

According to Eqs. (18) and (13),

$$\delta_{\mathbf{Q}}(\mathbf{B}) = \left( \frac{\prod_{j=1}^n \|\mathbf{b}_j\|}{\det \mathbf{R}} \right)^{1/n} \quad \mathbf{b}_j \stackrel{\text{def}}{=} \mathbf{R} \mathbf{e}_j \quad (19)$$

This relation shows that  $\delta_{\mathbf{Q}}(\mathbf{B})$  is the ‘dilute Euclidean-orthogonality defect’ of the basis  $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ . It can be shown that  $\det \mathbf{R}$  is the volume of the  $n$ -dimensional parallelepiped defined by these vectors. Clearly,  $\delta_{\mathbf{Q}}(\mathbf{B})$  is greater than or equal to 1, the zero defect corresponding to the case where  $\delta_{\mathbf{Q}}(\mathbf{B}) = 1$ .

The matrix  $\mathbf{M}$  whose columns are the column matrices  $\mathbf{e}_j$  of Eq. (18) is unimodular:  $\mathbf{M}$  is an integer  $n$ -by- $n$  matrix whose determinant is equal to  $\pm 1$ . The matrix relation

$$\mathbf{B} = \mathbf{B} \mathbf{M} \quad (20)$$

gathers the vector relations

$$\mathbf{e}_j = \sum_{i=1}^n m_{ij} \mathbf{e}_i \quad (\text{for } j = 1, \dots, n)$$

Clearly, the integers  $m_{i,j}$  are the entries of  $\mathbf{M}$ . In the same way as  $\mathbf{M}$  is associated with  $\mathbf{B}$ , the identity matrix  $\mathbf{I}_n$  is associated with  $\mathbf{B}$ . In terms of matrices, we have  $\mathbf{e}_j = \mathbf{M} \mathbf{e}_j$ , hence (from Eq. (18))

$$\delta_{\mathbf{Q}}(\mathbf{B}) = \left( \frac{\prod_{j=1}^n \mathbf{e}_j^T \mathbf{Q} \mathbf{e}_j}{\det \mathbf{Q}} \right)^{1/(2n)} \quad (21)$$

where

$$\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{M}^T \mathbf{Q} \mathbf{M} \quad (22)$$

Note that  $\det \mathbf{Q} = \det \mathbf{Q} = (\det \mathbf{R})^2$ . To compute  $\delta_{\mathbf{Q}}(\mathbf{B})$ , one is led to consider the factorization

$$\mathbf{Q} = \mathbf{U}^T \mathbf{D} \mathbf{U} \quad (23)$$

where  $\mathbf{U}$  is an upper-triangular matrix whose diagonal elements  $u_{j,j}$  are equal to unity;  $\delta_{\mathbf{Q}}(\mathbf{B})$  is then obtained via the logarithmic formula

$$\ln(\delta_{\mathbf{Q}}(\mathbf{B})) = \frac{1}{2n} \sum_{j=2}^n \ln \left( 1 + \sum_{i=1}^{j-1} \frac{d_i}{d_j} u_{i,j}^2 \right) \quad (24)$$

where the  $d_j$ ’s are the diagonal entries of  $\mathbf{D}$ . Note that

$$\ln(\delta_{\mathbf{Q}}(\mathbf{B})) = \frac{1}{2n} \sum_{j=2}^n \ln \left( 1 + \sum_{i=1}^{j-1} \frac{d_i}{d_j} u_{i,j}^2 \right) \quad (25)$$

As  $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{M}^T \mathbf{Q} \mathbf{M}$  (Eq. (22)),  $\delta_{\mathbf{Q}}(\mathbf{B})$  can also be regarded as the ‘reduction defect’ of  $\mathbf{Q}$  in basis  $\mathbf{B}$ , or in a more concise manner, as the reduction defect of  $\mathbf{Q}$ .

In what follows, the guiding idea is to choose  $\mathbf{M}$  so that  $\delta_{\mathbf{Q}}(\mathbf{B})$  be reduced somehow:  $\delta_{\mathbf{Q}}(\mathbf{B}) < \delta_{\mathbf{Q}}(\mathbf{B})$ . The notion of reduced basis introduced by Lenstra, Lenstra and Lovász (1982) was a key step in that direction.

### 3.4 LLL-reduced basis

DEFINITION 3.2. The column vectors  $\mathbf{e}_j$  of  $\mathbf{M}$  define an LLL-reduced basis of  $(\mathbb{Z}^n, \mathbf{Q})$  if the matrix elements of  $\mathbf{U}$  and  $\mathbf{D}$  in factorization (23) satisfy the conditions

$$|u_{i,j}| \leq \frac{1}{2} \quad \text{for } 1 \leq i < j \leq n \quad (26)$$

and

$$d_j \geq (\omega - u_{j-1,j}^2) d_{j-1} \quad \text{for } 2 \leq j \leq n \quad (27)$$

with  $1/4 < \omega < 1$   $\square$

Condition (26) reduces  $\delta_{\mathbf{Q}}(\mathbf{B})$  by reducing the size of the matrix elements  $u_{i,j}$ ; see Eqs. (25) and (24). Condition (27) requires the  $d_j$ 's be loosely sorted in increasing order with no distinctive discontinuity; the ratios  $d_i/d_j$  (for  $i < j$ ) are then made as small as 'LLL $_{\omega}$ -possible.'

### 3.5 Statement of the NLP problem in the reduced basis

To complete Sect. 3.1, we now state the NLP problem (9) in the selected reduced basis  $\mathbf{B}$ ; see the context of Eq (20). Clearly,  $\|\mathbf{v} - \hat{\mathbf{v}}\|_{\mathbf{Q}}^2 = \|\mathbf{M}[\mathbf{M}^{-1}(\mathbf{v} - \hat{\mathbf{v}})]\|_{\mathbf{Q}}^2$ . Setting

$$\mathbf{v} \stackrel{\text{def}}{=} \mathbf{M}^{-1}\mathbf{v} \quad \hat{\mathbf{v}} \stackrel{\text{def}}{=} \mathbf{M}^{-1}\hat{\mathbf{v}} \quad (28)$$

we therefore have

$$\begin{aligned} \|\mathbf{v} - \hat{\mathbf{v}}\|_{\mathbf{Q}}^2 &= \|\mathbf{M}(\mathbf{v} - \hat{\mathbf{v}})\|_{\mathbf{Q}}^2 \\ &= [\mathbf{v} - \hat{\mathbf{v}}]^T \mathbf{M}^T \mathbf{Q} \mathbf{M} [\mathbf{v} - \hat{\mathbf{v}}] \end{aligned}$$

It then follows that

$$\|\mathbf{v} - \hat{\mathbf{v}}\|_{\mathbf{Q}}^2 = q(\mathbf{v}) \quad (29)$$

where, from Eq. (23),

$$q(\mathbf{v}) \stackrel{\text{def}}{=} \|\mathbf{D}^{1/2} \mathbf{U}(\mathbf{v} - \hat{\mathbf{v}})\|^2 \quad (30)$$

Let  $\tilde{\mathbf{v}}$  now be a vector of  $\mathbb{Z}^n$  minimizing  $q(\mathbf{v})$ :

$$\tilde{\mathbf{v}} = \underset{\mathbf{v} \in \mathbb{Z}^n}{\text{argmin}} q(\mathbf{v}) \quad (31)$$

In the standard basis  $\mathbf{B}$ , the corresponding nearest lattice point is then obtained via the relation (see Eq. (28))

$$\tilde{\mathbf{v}} = \mathbf{M} \tilde{\mathbf{v}} \quad (32)$$

To tackle the optimization problem (31), it is convenient to introduce the vector  $\tilde{\mathbf{v}}$  defined via the relation

$$\mathbf{v} - \tilde{\mathbf{v}} \stackrel{\text{def}}{=} \mathbf{U}(\mathbf{v} - \hat{\mathbf{v}}) \quad (33)$$

As the diagonal elements of  $\mathbf{U}$  are equal to unity, the components of  $\tilde{\mathbf{v}}$ , the 'float conditioned ambiguities'  $\tilde{v}_j$ , are explicitly defined by the formula

$$\tilde{v}_j \stackrel{\text{def}}{=} \begin{cases} \hat{v}_n & \text{if } j = n \\ \hat{v}_j - \sum_{k=j+1}^n u_{j,k}(\hat{v}_k - \hat{v}_k) & \text{if } 1 \leq j < n \end{cases} \quad (34)$$

From Eqs. (30) and (33), we have

$$q(\mathbf{v}) = \sum_{j=1}^n d_j (\tilde{v}_j - \tilde{v}_j)^2 \quad (35)$$

The discrete-search methods presented in Sect. 5 derive from this equation.

## 4 LLL reduction

In Sects. 4.1 and 4.2, we introduce the reduction procedures that allow an LLL-reduced basis to be built; see Sect. 3.4. These procedures are basically involved in the LLL algorithm which provides all the related results. Our version of this algorithm, which derives from that of Luo and Qiao (2011), is presented in Sect. 4.3.

Throughout this section,  $\mathbf{D}$  and  $\mathbf{U}$  are the matrices of the factorization (23):  $\mathbf{Q} = \mathbf{U}^T \mathbf{D} \mathbf{U}$  for  $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{M}^T \mathbf{Q} \mathbf{M}$ ;  $\mathbf{M}$  is some unimodular matrix.

### 4.1 Procedure Reduce

If  $|u_{i,j}| > 1/2$  for some  $i < j$ , a procedure can be applied to ensure Condition (26). This procedure is referred to as REDUCE( $i, j$ ).

**Procedure R:** REDUCE( $i, j$ )

Consider the  $n$ -by- $n$  unimodular matrix

$$\mathbf{M}_{i,j} \stackrel{\text{def}}{=} \mathbf{I}_n - \lfloor u_{i,j} \rfloor \mathbf{e}_i \mathbf{e}_j^T \quad (i < j)$$

(Here,  $\mathbf{e}_i$  is the column matrix associated with the  $i$ th unit vector of  $\mathbf{B}$ .) Then, apply  $\mathbf{M}_{i,j}$  to  $\mathbf{U}$  and  $\mathbf{M}$  from the right-hand side:

$$\mathbf{U} := \mathbf{U} \mathbf{M}_{i,j} \quad \mathbf{M} := \mathbf{M} \mathbf{M}_{i,j} \quad \square$$

Only the elements of the  $j$ th columns of  $\mathbf{U}$  and  $\mathbf{M}$  can be affected by the action of  $\mathbf{M}_{i,j}$ :  $u_{i',j} := u_{i',j} - u_{i',i} \lfloor u_{i,j} \rfloor$  for all  $i'$ , and likewise  $m_{i',j} := m_{i',j} - m_{i',i} \lfloor u_{i,j} \rfloor$ . Concerning  $\mathbf{U}$ , as  $u_{i',j} = 0$  for  $i' > i$ , only the elements  $u_{i',j}$  for  $i' \leq i$  are affected. In particular,  $u_{i,j} := u_{i,j} - \lfloor u_{i,j} \rfloor$ . In the updated version of  $\mathbf{U}$ , we thus have  $|u_{i,j}| \leq 1/2$ .

### 4.2 Swap procedures

To ensure Condition (27), which is more subtle, some particular procedure is to be implemented. The core of the problem is then governed by the 2-by-2 matrices

$$\mathbf{D}_j \stackrel{\text{def}}{=} \begin{bmatrix} d_{j-1} & 0 \\ 0 & d_j \end{bmatrix} \quad (36)$$

and

$$\mathbf{U}_j \stackrel{\text{def}}{=} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \quad u \stackrel{\text{def}}{=} u_{j-1,j} \quad (37)$$

Setting (see procedure R)

$$\mathbf{M}_j^T \stackrel{\text{def}}{=} \begin{bmatrix} 1 & -\lfloor u \rfloor \\ 0 & 1 \end{bmatrix} \quad (38)$$

we have

$$\mathbf{U}_j \mathbf{M}_j^T = \begin{bmatrix} 1 & \check{u} \\ 0 & 1 \end{bmatrix} \quad \check{u} \stackrel{\text{def}}{=} u - \lfloor u \rfloor \quad (39)$$

Clearly,  $|\check{u}|$  is less than or equal to  $1/2$ .

Now, consider Condition (27) with  $u_{j-1,j} := \check{u}$ :

$$d_j \geq (\omega - \check{u}^2)d_{j-1}$$

When this condition is not satisfied, one is led to change the order of the corresponding ambiguity variables. We then say that

$$M_j \stackrel{\text{def}}{=} M_j^T S \quad \text{where} \quad S \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (40)$$

is a reduce-swap operator. From Eqs. (38) and (39), it follows that

$$M_j = \begin{bmatrix} -\lfloor u \rfloor & 1 \\ 1 & 0 \end{bmatrix} \quad U_j M_j = \begin{bmatrix} \check{u} & 1 \\ 1 & 0 \end{bmatrix} \quad (41)$$

Clearly,  $U_j M_j$  is not an upper-triangular matrix. Its original structure can be restored as specified in the following property. (The proof of this property is given in Appendix A.)

**Property RSR: REDUCESWAPRESTORE**

Matrix  $(U_j M_j)^T D_j (U_j M_j)$  can be factorized in the form

$$\bar{U}_j^T \bar{D}_j \bar{U}_j$$

where

$$\bar{D}_j \stackrel{\text{def}}{=} \begin{bmatrix} \bar{d}_{j-1} & 0 \\ 0 & \bar{d}_j \end{bmatrix} \quad \bar{U}_j \stackrel{\text{def}}{=} \begin{bmatrix} 1 & \bar{u} \\ 0 & 1 \end{bmatrix}$$

in which

$$\bar{d}_{j-1} \stackrel{\text{def}}{=} d_j + \check{u}^2 d_{j-1} \quad \bar{d}_j \stackrel{\text{def}}{=} d_j \frac{d_{j-1}}{\bar{d}_{j-1}} \quad \bar{u} \stackrel{\text{def}}{=} \check{u} \frac{d_{j-1}}{\bar{d}_{j-1}}$$

As a corollary,

$$G_j U_j M_j = \bar{U}_j \quad \text{where} \quad G_j \stackrel{\text{def}}{=} \begin{bmatrix} \bar{u} & 1 - \check{u}\bar{u} \\ 1 & -\check{u} \end{bmatrix}$$

$$\text{Moreover, } [G_j^{-1}]^T D_j G_j^{-1} = \bar{D}_j \quad \square$$

The following procedure in which  $u \stackrel{\text{def}}{=} u_{j-1,j}$  results from this property.

**Procedure RSR: REDUCESWAPRESTORE( $j$ )**

Compute  $\check{u} = u - \lfloor u \rfloor$ ,

$$\bar{d}_{j-1} = d_j + \check{u}^2 d_{j-1} \quad \bar{d}_j = d_j \frac{d_{j-1}}{\bar{d}_{j-1}} \quad \bar{u} = \check{u} \frac{d_{j-1}}{\bar{d}_{j-1}}$$

To update  $D$ , set  $\bar{d}_{j-1} := \bar{d}_{j-1}$  and  $\bar{d}_j := \bar{d}_j$ .

Then, for  $j \geq 2$ , let  $M_j \stackrel{\text{def}}{=} \text{diag}([\mathbf{I}_{j-2} \ M_j \ \mathbf{I}_{n-j}])$  be the matrix obtained from the identity matrix  $\mathbf{I}_n$  by substituting

$$M_j = \begin{bmatrix} -\lfloor u \rfloor & 1 \\ 1 & 0 \end{bmatrix}$$

for its 2-by-2 block with largest diagonal index  $j$ ; see Eq. (41). Likewise, define  $G_j \stackrel{\text{def}}{=} \text{diag}([\mathbf{I}_{j-2} \ G_j \ \mathbf{I}_{n-j}])$  where

$$G_j = \begin{bmatrix} \bar{u} & 1 - \check{u}\bar{u} \\ 1 & -\check{u} \end{bmatrix}$$

Matrices  $U$  and  $M$  are then updated as follows:

$$U := G_j U M_j \quad M := M M_j \quad \square$$

When implementing the operation  $G_j U M_j$ , the diagonal 2-by-2 block of  $U$  with largest diagonal index  $j$  is updated separately. Indeed, according to the corollary of Property RSR, it is equal to  $\bar{U}_j$ .

In the case where  $\lfloor u \rfloor = 0$ , this procedure reduces to Procedure SR: SWAPRESTORE( $j$ ).

### 4.3 LLL-type algorithms

The original LLL algorithm provides the matrices  $U$  and  $D$  involved in the LLL-reduced version of  $Q$  (see Eqs. (23) and (22)):

$$Q = U^T D U \quad \text{for} \quad Q \stackrel{\text{def}}{=} M^T Q M$$

It also yields the LLL-reduced basis  $B \stackrel{\text{def}}{=} B M$ ; see Sects. 3.3 and 3.4. Its main instructions are the following (see Eq. (16) for its initialization).

#### Original LLL algorithm

```

1   $U := \mathbf{U}; D := \mathbf{D}; M := \mathbf{I}_n$ 
2   $j := 2$ 
3  while  $j \leq n$ 
4      if  $|u_{j-1,j}| > 1/2$ , REDUCE( $j-1, j$ )
5      if  $d_j < (\omega - u_{j-1,j}^2)d_{j-1}$ 
6          SWAPRESTORE( $j$ )
7           $j := \max(j-1, 2)$ 
8      else
9          for  $i := j-2$  down to 1
10             if  $|u_{i,j}| > 1/2$ , REDUCE( $i, j$ )
11         endfor 9
12          $j := j+1$ 
13     endif 5
14 endwhile 3
```

Recently, Luo & Qiao (2011) proposed a modified LLL algorithm which can save a significant amount of operations, and also provides a basis for a parallel implementation. In that approach, which is justified via an example presented in Sect. 3 of their paper, the procedures imposing condition (26) are implemented at the end of this algorithm, once the LLL condition (27) has been imposed.

**LLL algorithm with delayed size-reduction**

```

1   $U := \mathbf{U}; D := \mathbf{D}; M := \mathbf{I}_n$ 
2   $j := 2$ 
3  while  $j \leq n$  [to impose Condition (27)]
4       $u := u_{j-1,j}$ 
5      if  $|u| > 1/2$ 
6          ReduceOption := true
7           $\tilde{u} := u - \lfloor u \rfloor$ 
8      else
9          ReduceOption := false
10          $\tilde{u} := u$ 
11     endif 5
12     if  $d_j < (\omega - \tilde{u}^2)d_{j-1}$ 
13         if ReduceOption = true
14             REDUCESWAPRESTORE( $j$ )
15         else
16             SWAPRESTORE( $j$ )
17         endif 13
18          $j := \max(j - 1, 2)$ 
19     else
20          $j := j + 1$ 
21     endif 12
22 endwhile 3
23 for  $j := 2 : n$  [to impose Condition (26)]
24     for  $i := j - 1$  down to 1
25         if  $|u_{i,j}| > 1/2$ 
26             REDUCE( $i, j$ )
27         endif
28     endfor 24
29 endfor 23

```

Typically, this LLL algorithm with ‘delayed size-reduction’ runs twice as fast as the original LLL algorithm. Compared to the algorithm of Luo and Qiao (2011), we made here the distinction between the procedures RSR and SR. Some CPU time can thus still be saved. Those changes concern the instruction blocks 5-11 and 13-17.

The procedures described in Sects. 4.1 and 4.2 can be completed so that this algorithm also provides the float solution in the LLL-reduced basis:  $\hat{v} = M^{-1}\hat{v}$ ; see Eq. (28). This can be done without forming  $M^{-1}$  explicitly.

According to Property RSR, we have

$$\bar{d}_{j-1} = d_j + \tilde{u}^2 d_{j-1}$$

Instruction 12 can therefore be equally well written in the form

$$12 \quad \text{if } \bar{d}_{j-1} < \omega d_{j-1}$$

At level  $j$ , the procedures RSR and SR modify, in particular, the matrix element  $u_{j-2,j-1}$ . As a result, this algo-

rithm has a ‘one-step up-and-down structure;’ see instructions 18 and 20. Lenstra, Lenstra and Lovász have shown that for any  $\omega$  in the open interval  $]1/4 \ 1[$ , the algorithm terminates: the number of times that the algorithm encounters the case where  $\bar{d}_{j-1} < \omega d_{j-1}$  is bounded. In the limit case where  $\omega = 1$ , the convergence can also be guaranteed; for further details, see Akhavi (2003), Nguyen and Stehlé (2009).

The convergence of the LLL algorithm is faster when reducing the value of the relaxation parameter  $\omega$ , but below some value (for example  $\omega = 0.70$ ), the dilute  $\mathbf{Q}$ -orthogonality defect of the LLL-reduced basis  $\mathbf{B}$  thus obtained begins to increase. The choice of  $\omega$  therefore depends on the context.

For example, in GNSS, when handling a regional network in real-time with  $n = 168$  and  $\delta_{\mathbf{Q}}(\mathbf{B}) \simeq 6.62$ ,  $\omega$  may reasonably be set equal to 0.9;  $\delta_{\mathbf{Q}}(\mathbf{B})$  can then be reduced to 1.19 for example. One then has a good compromise between the CPU time required for finding the reduced basis, and that used for the discrete search; see Sect. 5. On our old computers, the CPU time used for that LLL-reduction was 0.075 second with our LLL-type algorithm, against 0.141 second with the original LLL algorithm. The LLL algorithm with delayed size-reduction effectively leads to a gain of the order of two.

For the statistical developments involved in the GNSS validation procedures, such as those of Verhagen and Teunissen (2006), the choice  $\omega = 1$  is preferable. Indeed, as the discrete search is performed many times in the same reduced basis, the latter must be as  $\mathbf{Q}$ -orthogonal as possible.

## 5 Discrete search

This section is essentially devoted to the solution of the NLP problem in the selected reduced basis; see Sects. 3.1, 3.4, 3.5, and 4.3. The problem is therefore to minimize  $q(\mathbf{v})$  for  $\mathbf{v}$  lying in  $\mathbb{Z}^n$ ; see Eqs. (31) and (35).

Once the integer ambiguities  $v_n, v_{n-1}, \dots, v_{i+1}$  have been conditioned somehow (see the example given below), Eq. (34) provides the float conditioned ambiguity  $\tilde{v}_j$ .

*Example: Babai point.* Let us concentrate on Eq. (35) where the  $d_j$ ’s are loosely sorted in increasing order with no distinctive discontinuity. To find a point  $\mathbf{v}$  for which  $q(\mathbf{v})$  is a priori small, one is led to perform the ‘bootstrapping’ recursive process described below. The point thus formed is the Babai point  $\mathbf{v}^B$  [Babai (1986)]:

Level  $n$ :

$$v_n^B = \lfloor \tilde{v}_n \rfloor \quad \text{where } \tilde{v}_n = \hat{v}$$

Level  $n - 1$ :

$$v_{n-1}^B = \lfloor \tilde{v}_{n-1} \rfloor \quad \text{where } \tilde{v}_{n-1} = \hat{v}_{n-1} - u_{n-1,n}(v_n^B - \hat{v}_n)$$

$\vdots$

Level 1:

$$v_1^B = \lfloor \tilde{v}_1 \rfloor \quad \text{where } \tilde{v}_1 = \hat{v}_1 - \sum_{k=2}^n u_{1,k}(v_k^B - \hat{v}_k)$$



The Babai point is often the solution of the NLP problem, but not necessarily. In any case however (as explicitly shown in this section), it is the ‘natural starting point’ for searching this solution  $\square$

### 5.1 Ambiguity conditioning at level $j$

In the general case, in the process of conditioning ambiguity  $v_j$ , we will use the following notation (see Eq. (35))

$$s_j \stackrel{\text{def}}{=} \sum_{i=j}^n d_i (v_i - \tilde{v}_i)^2 \quad (42)$$

where  $\tilde{v}_i$  is given by (see Eq. (34))

$$\tilde{v}_i = \begin{cases} \hat{v}_n & \text{if } i = n; \\ \hat{v}_i - \sum_{k=i+1}^n u_{i,k} (v_k - \hat{v}_k) & \text{if } 1 \leq i < n \end{cases}$$

Note that  $s_j = t_j + d_j (v_j - \tilde{v}_j)^2$  where

$$t_j \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } j = n; \\ s_{j+1} & \text{if } j < n \end{cases} \quad (43)$$

Let us now assume that the ambiguities  $v_n, v_{n-1}, \dots, v_{i+1}$  have already been conditioned. Denoting by  $\ell$  an integer candidate for  $v_j$ , we then set

$$s \equiv s_j^{(\ell)} \stackrel{\text{def}}{=} t_j + d_j (\ell - \tilde{v}_j)^2 \quad (44)$$

The first ambiguity value  $\ell$  to be considered at level  $j$  is then

$$m = \lfloor \tilde{v}_j \rfloor \quad (45)$$

Indeed,  $|\ell - \tilde{v}_j|$  and thereby  $s$  are then as small as possible. In the process of minimizing  $q(v)$ , one is led to consider values of  $\ell$  other than  $m$ . These integers,  $\ell_1, \ell_2, \dots, \ell_p, \dots$ , where  $\ell_1 = m$ , are then sorted so that the discrepancies  $|\ell_p - \tilde{v}_j|$  form an increasing sequence. The second integer to be considered is therefore  $m + 1$  or  $m - 1$ . Two cases are thus distinguished (see Schnorr & Euchner (1994)):

**Schnorr<sup>(+)</sup>:**  $m < \tilde{v}_j$ . Ambiguity  $v_j$  may then be conditioned at the successive terms of the Schnorr list<sup>(+)</sup>

$m, m + 1, m - 1, m + 2, m - 2, m + 3, \dots$

**Schnorr<sup>(-)</sup>:**  $m \geq \tilde{v}_j$ . Ambiguity  $v_j$  may then be conditioned at the successive terms of the Schnorr list<sup>(-)</sup>

$m, m - 1, m + 1, m - 2, m + 2, m - 3, \dots$

In our implementation of the related approach, we save CPU time in the computation of the successive values of  $(\ell_p - \tilde{v}_j)^2$ . When handling the ambiguities  $\ell$ , and  $\ell + 1$  or  $\ell - 1$ , the following ‘perturbation formulas’ are then used:

$$\begin{cases} [(\ell + 1) - \tilde{v}_j]^2 = w^2 + (1 + 2w) \\ [(\ell - 1) - \tilde{v}_j]^2 = w^2 + (1 - 2w) \end{cases} \quad w \stackrel{\text{def}}{=} \ell - \tilde{v}_j \quad (46)$$

The multiplication  $w^2 := w \times w$  is then performed only for  $\ell := m$ ; see Sect 5.2. Many multiplications can thus be

avoided. Note that the calculation of  $2w$  is then to be made in an optimal manner ( $2w$  is not necessarily computed as the sum  $w + w$ ).

In the implementation of our approach, we used object-orientated programming (OOP), and introduced a specific object referred to as **SL** (for Schnorr list). More precisely, at the beginning of our program, we instantiated an array of  $n$  such objects, one at each level  $j$ . We then added two ‘methods’ linked to this object: **INIT** and **NEXT**. The latter are described in the following section.

### 5.2 Methods **INIT** and **NEXT**

The actions of **INIT** and **NEXT** consist in initializing and updating a two-element FIFO vectorial queue  $(\ell_a, \ell_b)$ ,  $(s_a, s_b)$  associated with the two-component vector  $(\ell, s)$ . The table below shows the structure of queue  $(\ell_a, \ell_b)$  in the case of the Schnorr list<sup>(+)</sup>:

	sg	$\ell_a$	$\ell_b$
After <b>INIT</b> :	+1	$m$	$m$
After <b>NEXT</b> :	-1	$m$	$m + 1$
After <b>NEXT</b> :	+1	$m + 1$	$m - 1$
After <b>NEXT</b> :	-1	$m - 1$	$m + 2$

Just before the call to **INIT**,  $\tilde{v}_j$  is computed on the grounds of Eq. (34); see Remark 5.1 further on.

**Method **INIT**:** instruction  $(\ell, s) := \text{SL}_j\text{-INIT}(\tilde{v}_j, t_j)$

Set

$$\begin{aligned} \ell &:= \lfloor \tilde{v}_j \rfloor \\ w &:= \ell - \tilde{v}_j \\ s &:= t_j + d_j w^2 \\ \ell_a &:= \ell_b := \ell \\ s_a &:= s_b := s \end{aligned}$$

if  $w < 0$

$$\text{set sg} := (+1)$$

else

$$\text{set sg} := (-1)$$

**Method **NEXT**:** instruction  $(\ell, s) := \text{SL}_j\text{-NEXT}$

Set

$$\begin{aligned} w &:= \ell_a - \tilde{v}_j \\ \ell &:= \ell_a + \text{sg} \end{aligned}$$

if  $\text{sg} = 1$

$$s := s_a + d_j (1 + 2w)$$

else

$$s := s_a + d_j (1 - 2w)$$

Set

$$\ell_a := \ell_b; \ell_b := \ell$$

$$s_a := s_b; s_b := s$$

$$\text{sg} := (-\text{sg})$$

**Remark 5.1.** According to Eq. (34), the float conditioned ambiguity  $\tilde{v}_j$  is given by the formula

$$\tilde{v}_j = \begin{cases} \hat{v}_n & \text{if } j = n \\ \tilde{u}_{j,j+1} & \text{if } 1 \leq j < n \end{cases} \quad (47)$$

where

$$\tilde{u}_{j,k} \stackrel{\text{def}}{=} \hat{v}_j - \sum_{\kappa=k}^n u_{j,\kappa} (v_\kappa - \hat{v}_\kappa) \quad (48)$$

Now, consider the general case when  $\tilde{v}_j$  is to be computed, when it has already been computed, and when in the meanwhile, for some  $j_r > j$ , the integer ambiguities  $v_{j_r+1}, v_{j_r+2}, \dots, v_{n-1}, v_n$  have not changed. In our conditioning process, to reduce the corresponding CPU cost,  $\tilde{v}_j$  is then computed as follows (see Eqs. (47) and (48)):

```

If  $j_r = n$  (even if  $\tilde{v}_j$  has not been computed yet)
     $u := \hat{v}_j$ 
else
     $u := \tilde{u}_{j,j_r+1}$ 
for  $k := j_r$  down to  $k := j + 1$ 
     $u := u - u_{j,k} (v_k - \hat{v}_k)$ 
     $\tilde{u}_{j,k} := u$ 
endfor
 $\tilde{v}_j := u$ 

```

An auxiliary upper-triangular matrix  $\tilde{U}$  is thus built and updated through the process. For further details, see Sect. 5.3 and Remark 5.3  $\square$

### 5.3 Discrete-search algorithms

On the grounds of the notions introduced in Sects. 5.1 and 5.2, we have designed three discrete-search algorithms referred to as DS, DNS and DSC:

- 1) algorithm DS yields a nearest lattice point  $\tilde{v}$  and  $\tilde{q} \stackrel{\text{def}}{=} q(\tilde{v})$ ;
- 2) algorithm DNS provides the first  $n_s$  NLP solutions  $\tilde{v}_1 \equiv \tilde{v}, \tilde{v}_2, \dots, \tilde{v}_{n_s}$  with  $\tilde{q} \equiv \tilde{q}_1 \leq \tilde{q}_2 \leq \dots \leq \tilde{q}_{n_s}$ ;
- 3) given some parameter  $c > 0$ , algorithm DSC identifies all the points  $v$  of  $\mathbb{Z}^n$  contained in the ellipsoid

$$\mathcal{E}(c) \stackrel{\text{def}}{=} \{v \in \mathbb{Z}^n : q(v) \leq c\} \quad (49)$$

Clearly,  $\mathcal{E}(c)$  is centred on the float solution  $\hat{v}$ ;  $c$  defines the size of this ellipsoid.

**Algorithm DS.** The objective is to condition the integer ambiguities  $v_j$  so that  $q(v)$  is minimum. We first note that from Eqs. (35) and (42),

$$\begin{aligned} q(v) &= s_1 \\ &= r_j + s_j \end{aligned} \quad (50)$$

where

$$r_j \stackrel{\text{def}}{=} \sum_{i=1}^{j-1} d_i (v_i - \tilde{v}_i)^2 \quad (51)$$

As  $r_j$  is non-negative, we therefore have:

**Property 5.1.** If  $s_j \geq a$  for some  $a > 0$ , then  $q(v) = s_1 \geq a$ .

We first form the Babai point, here  $v := v^B$ ; see the bootstrapping stage 2-8 of the algorithm displayed in the next page. All the Schnorr lists from  $j := n$  down to  $j := 1$ , as well as  $\tilde{U}$ , are thus initialized; see Remark 5.1 with  $j_r = n$ . As the Babai point is the first NLP candidate, we then set

$$\tilde{v} := v, \quad q(\tilde{v}) \equiv \tilde{q} := s_1$$

The NLP search starts from the Babai point, but in the opposite sense, with a Boolean variable Forwards equal to true. We therefore move to level  $j = 2$ . Indeed, if  $v_1$  was set equal to the next integer of  $\text{SL}_1$ ,  $q(v)$  would then be greater than  $\tilde{q}$ .

To understand the principle of the algorithm in the general case, let us assume that we are at some level  $j \geq 2$  with Forwards = true. We then consider the integer  $\ell$  provided by  $\text{SL}_j\text{-NEXT}$ ; this method also yields  $s$ : the new value of  $s_j$  that would be obtained if  $v_j$  was set equal to  $\ell$ . Clearly,  $s$  is greater than the current value of  $s_j$  (and this would be worse with the remaining terms of the Schnorr list at this level). Two cases are then to be considered.

**Case 1:**  $s \geq \tilde{q}$ . If we then set  $v_j := \ell$ , whatever the conditioning of the integer ambiguities  $v_{j-1}, \dots, v_1$ , we would then have  $s_1 \geq \tilde{q}$  from Property 5.1. Furthermore, another  $\text{NEXT}$ -type instruction would increase  $s_j$ . In this case, we are therefore left to move forwards to level  $j := j + 1$ .

**Case 2:**  $s < \tilde{q}$ . As there is still a hope of reducing  $s_1$  by conditioning  $v_{j-1}, \dots, v_1$  in an appropriate manner, we then set

$$(v_j, s_j) := (\ell, s), \quad t_{j-1} := s_j, \quad \text{Forwards} := \text{false}$$

and move backwards to level  $j := j - 1$ ;  $\tilde{v}_j$  is then updated; note that  $(\lfloor \tilde{v}_j \rfloor - \tilde{v}_j)^2$  may then be smaller than previously at that level.

When the algorithm moves forwards to level  $j := j + 1$ ,  $\text{SL}_j\text{-NEXT}$  is then called. When it moves backwards to level  $j := j - 1$ , a new Schnorr list is initialized via  $\text{SL}_j\text{-INIT}$ . In both cases, the situation is then analysed to define what is to be done; see Cases 1 and 2.

Via Case 2, the algorithm may progressively reach level  $j = 1$  (several times). If  $s$  is less than  $\tilde{q}$ ,  $\tilde{v}$  and  $\tilde{q}$  are then updated; see instructions 32 to 36.

Via Case 1, the algorithm reaches level  $n$ , at least once. When  $\text{SL}_n\text{-NEXT}$  yields an  $s$  greater than or equal to  $\tilde{q}$ , the algorithm then stops; see instructions 14 to 25. We then have the following property (see Eq. (24)):

**Property 5.2.** At the end of the algorithm, no point of  $\mathbb{Z}^n$  lies in the interior of ellipsoid  $\mathcal{E}(\tilde{q})$ ;  $\tilde{v}$  is on its boundary.

**Algorithm DS**

```

1   $t_n := 0; j_r := n$ 
2  for  $j := n$  down to  $j := 1$     [Babai loop]
3      Compute  $\tilde{v}_j$ 
4       $(\ell, s) := \text{SL}_j\text{-INIT}(\tilde{v}_j, t_j)$ 
5       $(v_j, s_j) := (\ell, s)$ 
6      if  $j > 1$  set  $t_{j-1} := s_j$ 
7  endfor 2
8   $(\tilde{\mathbf{v}}, \tilde{q}) := (\mathbf{v}, s_1)$     [Babai point]
9  NLPfound := false
10 Forwards := true
11  $j_1 := 1; j_2^* := 1$ 
12  $j := 1$ 
13 while NLPfound = false    [NLP search]
14     if Forwards = true    [move forwards]
15         if  $j = n$ 
16             NLPfound := true
17         else
18              $j := j + 1$ 
19              $(\ell, s) := \text{SL}_j\text{-NEXT}$ 
20             if  $s < \tilde{q}$ 
21                  $(v_j, s_j) := (\ell, s); t_{j-1} := s_j$ 
22                 Forwards := false
23                  $j_2 := j; j_2^* := \max(j_2, j_2^*)$ 
24             endif 20
25         endif 15
26     else    [move backwards]
27          $j := j - 1$ 
28         if  $j < j_1$  set  $j_r := j_2^*$ 
29         else set  $j_r := j_2$ 
30         Compute  $\tilde{v}_j$ 
31          $(\ell, s) := \text{SL}_j\text{-INIT}(\tilde{v}_j, t_j)$ 
32         if  $j = 1$  [case  $j = 1$ ]
33             if  $s < \tilde{q}$ 
34                  $(v_1, s_1) := (\ell, s)$ 
35                  $(\tilde{\mathbf{v}}, \tilde{q}) := (\mathbf{v}, s_1)$     [new  $\tilde{\mathbf{v}}$ ]
36             endif 33
37             Forwards := true
38              $j_1 := 1; j_2^* := 1$ 
39         else    [case  $j > 1$ ]
40             if  $s < \tilde{q}$ 
41                  $(v_j, s_j) := (\ell, s); t_{j-1} := s_j$ 
42             else
43                 Forwards := true
44                  $j_1 := j$ 
45             endif 40
46         endif 32
47     endif 14
48 endwhile 13

```

*Proof.* Let us assume that there exists some  $\mathbf{v}^\circ$  in  $\mathbb{Z}^n$  such that  $q^\circ \stackrel{\text{def}}{=} q(\mathbf{v}^\circ) < \tilde{q}$ . From Eqs. (35) and (34),

$$q^\circ = d_n(v_n^\circ - \hat{v}_n)^2 + d_{n-1}(v_{n-1}^\circ - \tilde{v}_{n-1}^\circ)^2 + \cdots + d_1(v_1^\circ - \tilde{v}_1^\circ)^2$$

The quantities

$$s_n^\circ \stackrel{\text{def}}{=} d_n(v_n^\circ - \hat{v}_n)^2$$

and

$$s_j^\circ \stackrel{\text{def}}{=} s_n^\circ + d_{n-1}(v_{n-1}^\circ - \tilde{v}_{n-1}^\circ)^2 + \cdots + d_j(v_j^\circ - \tilde{v}_j^\circ)^2 \quad (1 \leq j < n)$$

are then less than  $\tilde{q}$ . The algorithm starts by setting  $v_n$  equal to the first term of the Schnorr list  $\text{SL}_n$ . In the NLP search, it then comes back to level  $n$  via instruction 18, at least once, until  $v_n$  is conditioned at  $v_n^\circ$ ; indeed,  $s_n^\circ$  is less than  $\tilde{q}$  (see instructions 19 to 21). The algorithm then starts moving backwards (via instruction 22), and reaches instruction 31 with  $j = n - 1$ . The Schnorr list  $\text{SL}_{n-1}$  is then systematically explored, with possible excursions at levels  $j < n - 1$ , and this until  $v_{n-1}$  is set equal to  $v_{n-1}^\circ$ , since  $s_{n-1}^\circ < \tilde{q}$ . The algorithm then proceeds to level  $n - 2$ . It then behaves similarly for that level, and so on, until level  $j = 1$  where  $v_1$  is set equal to the first term of  $\text{SL}_1$ . The condition  $s < \tilde{q}$  of instruction 33 then holds, hence via instructions 34 and 35,  $\tilde{q} := s \leq q^\circ$ , which contradicts the initial assumption.  $\square$

*Remark 5.2.* The pathological situations where  $\tilde{\mathbf{v}}$  is not the only point of  $\mathbb{Z}^n$  on the boundary of  $\mathcal{E}(\tilde{q})$  can be detected at level of instruction 33. The integer-ambiguity solution  $\tilde{\mathbf{v}}$  cannot then be validated. As it is presented, the algorithm selects as solution the first  $\mathbf{v}$  for which  $q(\mathbf{v}) = \tilde{q}$ ; the other ones (if any) are discarded. A subsequent statistical analysis can be used to diagnose such pathological cases. In practice, as expected, such situations never occur  $\square$

*Remark 5.3.* In the NLP search (instructions 13 to 48), the integers  $j_1$  and  $j_2$  keep track of the successive levels  $j$  at which the value of the Boolean variable Forwards changes. Note that  $j_2^*$  is the current largest index  $j_2$  at which the algorithm started moving backwards. According to instructions 11 and 38, whenever  $j = 1$ ,  $j_1$  and  $j_2^*$  are set equal to 1. By computing the float conditioned ambiguities in the framework of Remark 5.1 in which  $j_r$  is defined (from  $j_1$ ,  $j_2$  and  $j_2^*$ ) via instructions 28-29, the global CPU time of algorithm DS can be reduced by a factor of the order of two. In this context, the following technical point is also to be mentioned.

First of all, at the beginning of algorithm DS, the values of  $\hat{v}_j$  are placed on the diagonal of  $\tilde{\mathbf{U}}$ :

$$\tilde{u}_{j,j} := \hat{v}_j \quad (\text{for } j := 1, \dots, n)$$

Instructions 5, 21 and 41 are then completed by setting

$$v_j^* := \ell - \hat{v}_j$$

The instructions  $u := \hat{v}_j$  and  $u := u - u_{j,k}(v_k - \hat{v}_k)$  of Remark 5.1 are then changed into  $u := \tilde{u}_{j,j}$  and  $u := u - u_{j,k}v_k^*$ , respectively. The input variables of the function that computes  $\tilde{v}_j$  are then  $j, j_r, n, U, \tilde{U}$  and  $v^*$   $\square$

**Remark 5.4.** At the beginning of the NLP search, the size parameter  $c$  of the search ellipsoid (49) is defined by the value of  $q(v)$  for the Babai point. When the latter is not the NLP solution,  $c \equiv \tilde{q}$  is reduced via instruction 35  $\square$

**Algorithm DNS.** The process is similar to that of algorithm DS; but, once the Babai point has been formed, instead of moving forwards to level  $j = 2$ , **SL<sub>1</sub>-NEXT** is set in motion  $n_s - 1$  times. We thus get a ‘working set’ including  $n_s$  ‘candidate optimal lattice points’  $\tilde{v}[n_s]$  together with their  $q$ -values  $\tilde{q}[n_s]$ . The last  $q$ -value thus obtained, which (by construction) is larger than the previous ones, is denoted by  $\tilde{q}_{n_s}$ . In algorithm DNS,  $\tilde{q}_{n_s}$  is going to play the same role as  $\tilde{q}$  in algorithm DS.

We then move forwards to level  $j = 2$ ; **SL<sub>2</sub>-NEXT** then provides the next term  $\ell$  of the Schnorr list at level 2 together with the value of  $s$  for that  $\ell$ . If  $s$  is less than  $\tilde{q}_{n_s}$ , we then set  $(v_2, s_2) := (\ell, s)$ ,  $t_1 := s_2$ , and move backwards to level 1; **SL<sub>1</sub>-INIT** then defines (via  $\ell$ ) some lattice point  $v$  with its  $q$ -value  $q(v) := s_1 := s$ . If  $s$  is less than  $\tilde{q}_{n_s}$ , as  $v$  does not lie in the current set  $\tilde{v}[n_s]$ ,  $s$  and  $v$  have to be inserted at their right places in the sets  $\tilde{q}[n_s]$  and  $\tilde{v}[n_s]$ ; the previous  $\tilde{q}_{n_s}$  and  $\tilde{v}_{n_s}$  are then removed. Instruction **SL<sub>j</sub>-NEXT** is then performed until  $s$  is larger than the current value of  $\tilde{q}_{n_s}$ . After each of these instructions,  $\tilde{q}[n_s]$  and  $\tilde{v}[n_s]$  are of course updated and sorted. In any case, we then finally move forwards to level  $j = 2$ ; **SL<sub>2</sub>-NEXT** is then performed, and so on. Clearly, the principle is the same.

**Algorithm DSC.** The process is again similar to that of algorithm DS. As all the points of ellipsoid  $\mathcal{E}(c)$  are to be identified, the tests  $s < \tilde{q}$  (the instructions 20, 33 and 40 of algorithm DS) are replaced by  $s < c$ . When **SL<sub>j</sub>-NEXT** is called, we move forwards to level  $j + 1$ , only when the value of  $s$  thus obtained is larger than (or equal to)  $c$ ; see Eq. (49) and Property 5.1. Otherwise, we set  $(v_j, s_j) := (\ell, s)$ ,  $t_{j-1} := s_j$  and move backwards:  $j := j - 1$ ; then **SL<sub>j</sub>-INIT**, and so on. Instruction 35 of algorithm DS is replaced by other instructions which depend on what is to be done with the vector  $v$  thus identified; see, e.g., Verhagen and Teunissen (2006), Lannes and Prieur (2011).

## 6 On some computational issues

The serial algorithms presented in Sects. 4.3 and 5.3 were implemented in C++ programs, and tested on old PC's working with Windows XP and Linux operating systems. Intensive testing was performed with real data on a regional GNSS network. As already mentioned at the end of Sect. 4.3, for  $n = 168$ , the CPU time for the execution of our LLL-type algorithm with  $\omega = 0.9$  was negligible: about 0.075 second. Compared to the original LLL algorithm, as

implemented for instance by Agrell et al. (2002) or Jazaeri et al. (2012), the gain was of the order of two. In fact, the parallel approach begins to be of interest for  $n$  larger than (say) 200; see the reduction-list implementation of Luo and Qiao (2011).

Concerning the discrete-search algorithms presented in this paper, our method was compared to that of Jazaeri et al. (2012) which corresponds to the present state of the art for the discrete search. Our statistical study on  $3 \times 10^5$  Gaussian  $\hat{v}$ -samples was conducted for  $n = 168$  in the LLL-reduced basis obtained as already specified. The Gaussian  $\hat{v}$ -samples were of mean  $\mathbf{0}$  and variance-covariance matrix  $V = Q^{-1}$  in that basis. For each sample,  $\tilde{v}_1 \equiv \tilde{v}$  and  $\tilde{v}_2$  were determined via our DNS algorithm; see Sect. 5.3. The CPU times for those discrete searches were 236 seconds with the algorithm of Jazaeri et al. (2012), and 129 seconds with our DNS algorithm. This gain, which is of the order of two, is essentially due to the way of computing the float conditioned ambiguities; see Remarks 5.1 and 5.3.

With regard to the self-calibration problems presented in Sect. 2.1, the previous statistical study gives an idea of the efficiency of our methods for finding the global and secondary minima of the arc functional  $g$ ; see Sect 2.1.

For handling the Schnorr lists at best, some object-oriented programming tools have been introduced; see Sect. 5.2. Our discrete-search algorithms were thereby written in an ‘almost-electronic form.’ Shortly, they were designed for DSP (digital signal processor) implementation at the ‘speed of light.’ In GNSS, for example, the integer ambiguities of regional networks can thus be fixed in real time. Let us finally note that for large  $n$ , the only discrete-search operations that can be performed in a parallel manner are those associated with the successive terms of the Schnorr lists at levels  $n$  and 1.

## 7 Conclusion

In this paper, we presented new methods for solving the nearest-lattice point (NLP) problems arising in astronomy, geodesy and GNSS. The main theoretical aspects of the matter were also analysed. This contribution concerns both the preconditioning stage, and the discrete-search stage in which the integer ambiguities are finally fixed. We proposed several algorithms whose efficiency was shown via intensive numerical tests on GNSS data. The same algorithms can be used in the astronomical self-calibration procedures. The related NLP problems are indeed very similar.

Concerning the preconditioning stage, we have shown that the LLL-type algorithms with delayed size-reduction lead to a gain of the order of two relative to the standard LLL algorithm. We have particularly optimized the discrete-search (DS) algorithms. Our DS algorithms run also about twice as fast as the state-of-the-art DS algorithms of Jazaeri et al. (2012). We have thus been able to perform intensive calculations on large-size problems with our old computers. This would be particularly interesting for real-time data

processing of world-wide global GNSS networks. As explicitly shown in Lannes (2013), parallel versions of our LLL-type algorithms could also be implemented for those extreme cases.

In astronomy, our self-calibration approach could lead to a substantial gain in computing time for large interferometric arrays. Another important asset of our approach is to propose a method for validating the calibration solution. For each phase-calibration operation, we determine the global minimum of the arc functional and the first secondary minima (if any); see Sects. 2.1 and 5.3 in this paper, and Sect. 5 in Lannes & Prieur 2011. In the case of multiple minima, the relative discrepancy between the values the global and secondary minima provides a measure against which the reliability of the process can be appreciated. This is an innovative approach which could promote the use of the self-calibration procedures in radio imaging. In particular, the extension of our approach to matrix self-calibration is an interesting problem that we intend to address in a forthcoming paper.

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## Appendix A Proof of Property RSR

The proof of Property REDUCESWAPRESTORE can be obtained as follows.

From Eqs. (41) and (36), we have

$$(U_j M_j)^T D_j (U_j M_j) = \begin{bmatrix} \check{u} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{j-1} & 0 \\ 0 & d_j \end{bmatrix} \begin{bmatrix} \check{u} & 1 \\ 1 & 0 \end{bmatrix}$$

i.e., explicitly,

$$(U_j M_j)^T D_j (U_j M_j) = \begin{bmatrix} \bar{d}_{j-1} & d_{j-1} \check{u} \\ d_{j-1} \check{u} & d_{j-1} \end{bmatrix}$$

Let us now factorize this matrix in the form

$$\begin{aligned} U^T D U &= \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} \begin{bmatrix} c_{j-1} & 0 \\ 0 & c_j \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{j-1} & c_{j-1} u \\ c_{j-1} u & c_j + c_{j-1} u^2 \end{bmatrix} \end{aligned}$$

By identifying the corresponding terms, we have

$$c_{j-1} = \bar{d}_{j-1} \quad c_{j-1} u = d_{j-1} \check{u} \quad c_j + c_{j-1} u^2 = d_{j-1}$$

As a result,  $u = \bar{u}$  and  $c_j + \bar{d}_{j-1} u^2 = d_{j-1}$ , hence

$$\begin{aligned} c_j &= d_{j-1} - \bar{d}_{j-1} \check{u}^2 \frac{d_{j-1}^2}{\bar{d}_{j-1}^2} \\ &= d_{j-1} \left( 1 - \check{u}^2 \frac{d_{j-1}}{\bar{d}_{j-1}} \right) \\ &= \frac{d_{j-1}}{\bar{d}_{j-1}} (\bar{d}_{j-1} - \check{u}^2 d_{j-1}) \\ &= \frac{d_{j-1}}{\bar{d}_{j-1}} d_j \\ &= \bar{d}_j \end{aligned}$$

Consequently,  $(U_j M_j)^T D_j (U_j M_j) = \bar{U}_j^T \bar{D}_j \bar{U}_j$ .

The corollary results from the fact that (see Eq. (41))

$$\bar{U}_j (U_j M_j)^{-1} = \begin{bmatrix} 1 & \bar{u} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\check{u} \end{bmatrix} = \begin{bmatrix} \bar{u} & 1 - \check{u} \bar{u} \\ 1 & -\check{u} \end{bmatrix}$$

i.e.,  $\bar{U}_j (U_j M_j)^{-1} = G_j$ , hence  $G_j U_j M_j = \bar{U}_j$ . We then have

$$\begin{aligned} (U_j M_j)^T D_j (U_j M_j) &= (G_j U_j M_j)^T (G_j^{-1})^T D_j G_j^{-1} (G_j U_j M_j) \\ &= \bar{U}_j^T \bar{D}_j \bar{U}_j \end{aligned}$$

hence  $(G_j^{-1})^T D_j G_j^{-1} = \bar{D}_j \quad \square$